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HIGH-PRECISION CHEBYSHEV SERIES APPROXIMATION TO THE EXPONENTIAL INTEGRAL

by Kin L. Lee

Ames Research Center
Moffett Field, Calif. 94035

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16. Abstract

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HIGH-PRECISION CHEBYSHEV SERIES APPROXIMATION TO THE EXPONENTIAL INTEGRAL

Kin L. Lee

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SUMMARY

The exponential integral Ei(x) is evaluated via Chebyshev series expansion of its associated functions to achieve high relative accuracy throughout the entire real line. The Chebyshev coefficients for these functions are given to 30 significant digits. Clenshaw's method is modified to furnish an efficient procedure for the accurate solution of linear systems having near-triangular coefficient matrices.

INTRODUCTION

The evaluation of the exponential integral

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^u}{u}\, du = -E_1(-x) , \qquad x \neq 0 \tag{1}$$

is usually based on the value of its associated functions, for example, $xe^{-x}\mathrm{Ei}(x)$. High accuracy tabulations of integral (1) by means of Taylor series techniques are given by Harris (ref. 1) and Miller and Hurst (ref. 2). The evaluation of Ei(x) for $-4 \leq x < \infty$ by means of Chebyshev series is provided by Clenshaw (ref. 3) to have the absolute accuracy of 20 decimal places. The evaluation of the same integral (1) by rational approximation of its associated functions is furnished by Cody and Thacher (refs. 4 and 5) for $-\infty < x < \infty$, and has the relative accuracy of 17 significant figures.

The approximations of Cody and Thacher from the point of view of efficient function evaluation are preferable to those of Clenshaw. However, the accuracy of the latter's procedure, unlike that of the former, is not limited by the accuracy or the availability of a master function, which is a means of explicitly evaluating the function in question.

In this paper Ei(x) (or equivalently $-E_1(-x)$) for the entire real line is evaluated via Chebyshev series expansion of its associated functions that are accurate to 30 significant figures by a modification of Clenshaw's procedure. To verify the accuracy of the several Chebyshev series, values of the associated functions were checked against those computed by Taylor series and those of Murnaghan and Wrench (ref. 6) (see Remarks on Convergence and Accuracy).

Although for most purposes fewer than 30 figures of accuracy are required, such high accuracy is desirable for the following reasons. In order to further reduce the number of arithmetical operations in the evaluation of a function, the Chebyshev series in question can either be converted into a rational function or rearranged into an ordinary polynomial. Since several figures may be lost in either of these procedures, it is necessary to provide the Chebyshev series with a sufficient number of figures to achieve the desired accuracy. Furthermore, general function approximation routines, such as those used for minimax rational function approximations, require the explicit evaluation of the function to be approximated. To take account of the errors committed by these routines, the function values must have an accuracy higher than the approximation to be determined. Consequently, high-precision results are useful as a master function for finding approximations for (or involving) Ei(x) (e.g., refs. 4 or 5) where prescribed accuracy is less than 30 figures.

DISCUSSION

It is proposed here to provide for the evaluation of Ei(x) by obtaining Chebyshev coefficients for the associated functions given in table 1.

TABLE 1.- ASSOCIATED FUNCTIONS OF Ei(x) AND THEIR RANGES OF CHEBYSHEV SERIES EXPANSION

Associated function	Range of expansion
a. $xe^{-x}\mathrm{Ei}(x)$	$-\infty < x \leq -10$
b. $xe^{-x}\mathrm{Ei}(x)$	$-10 \leq x \leq -4$
c. $\dfrac{\mathrm{Ei}(x) - \log\lvert x\rvert - \gamma}{x}$	$-4 \leq x \leq 4$
d. $xe^{-x}\mathrm{Ei}(x)$	$4 \leq x \leq 12$
e. $xe^{-x}\mathrm{Ei}(x)$	$12 \leq x \leq 32$
f. $xe^{-x}\mathrm{Ei}(x)$	$32 \leq x < \infty$

(γ = 0.5772156649... is Euler's constant.)

Note that the functions $[\mathrm{Ei}(x) - \log\lvert x\rvert - \gamma]/x$ and $xe^{-x}\mathrm{Ei}(x)$ have the limiting values of unity at the origin and at infinity, respectively, and that the range of the associated function values is close to unity (see table 4). This makes for the evaluation of the associated functions over the indicated ranges in table 1 (and thus Ei(x) over the entire real line) with high relative accuracy by means of the Chebyshev series. The reason for this will become apparent later.

Some remarks about the choice of the intervals of expansion for the several Chebyshev series are in order here. The partition of the real line indicated in table 1 is chosen to allow for the approximation of the associated functions with a maximum error of 0.5×10^{-30} by polynomials of degree < 50. The real line has also been partitioned with the objective of providing the interval about zero with the lowest degree of polynomial approximation of the six intervals. This should compensate for the computation of $\log |x|$ required in the evaluation of $Ei(x)$ over that interval. The ranges $-\infty < x \le -4$ and $4 \le x < \infty$ are partitioned into 2 and 3 intervals, respectively, to provide approximations to $xe^{-x}Ei(x)$ by polynomials of about the same degree.

Expansions in Chebyshev Series

Let $\phi(t)$ be a differentiable function defined on $[-1, 1]$. To facilitate discussion, denote its Chebyshev series and that of its derivative by

$$\phi(t) = \sum_{k=0}^{\infty}{}' A_k^{(0)} T_k(t) , \qquad \phi'(t) = \sum_{k=0}^{\infty}{}' A_k^{(1)} T_k(t) \tag{2}$$

where $T_k(t)$ are Chebyshev polynomials defined by

$$T_k(t) = \cos(k \arccos t) , \qquad -1 \le t \le 1 \tag{3}$$

(A prime over a summation sign indicates that the first term is to be halved.)

If $\phi(t)$ and $\phi'(t)$ are continuous, the Chebyshev coefficients $A_k^{(0)}$ and $A_k^{(1)}$ can be obtained analytically (if possible) or by numerical quadrature. However, since each function in table 1 satisfies a linear differential equation with polynomial coefficients, the Chebyshev coefficients can be more readily evaluated by the method of Clenshaw (ref. 7).

There are several variations of Clenshaw's procedure (see, e.g., ref. 8), but for high-precision computation, where multiple precision arithmetic is employed, we find his original procedure easiest to implement. However, straightforward application of it may result in a loss of accuracy if the trial solutions selected are not sufficiently independent. How the difficulty is overcome will be pointed out subsequently.

The Function $xe^{-x}Ei(x)$ on the Finite Interval

We consider first the Chebyshev series expansion of

$$f(x) = xe^{-x}Ei(x) , \qquad (a \le x \le b) \tag{4}$$

with $x \neq 0$. One can easily verify that after the change of variables

$$x = [(b - a)t + a + b]/2 , \qquad -1 \leq t \leq 1 \tag{5}$$

the function

$$\phi(t) = f\left[\frac{(b - a)t + a + b}{2}\right] = f(x) \tag{6}$$

satisfies the differential equation

$$2(pt + q)\phi'(t) + p(pt + q - 2)\phi(t) = p(pt + q) \tag{7a}$$

with[1]

$$\phi(-1) = ae^{-a}Ei(a) \tag{7b}$$

where $p = b - a$ and $q = b + a$. Replacing $\phi(t)$ and $\phi'(t)$ in equations (7) by their Chebyshev series, we obtain

$$\sum_{k=0}^{\infty}{}' (-1)^k A_k^{(0)} = \phi(-1) \tag{8a}$$

$$2\sum_{k=0}^{\infty}{}' A_k^{(1)}(pt + q)T_k(t) + p\sum_{k=0}^{\infty}{}' A_k^{(0)}(pt + q - 2)T_k(t) = p(pt + q) \tag{8b}$$

It can be demonstrated that if B_k are the Chebyshev coefficients of a function $\psi(t)$, then C_k, the Chebyshev coefficients of $t^r\psi(t)$ for positive integers r, are given by (ref. 7)

$$C_k = 2^{-r}\sum_{i=0}^{r}\binom{r}{i} B_{|k-r+2i|} \tag{9}$$

Consequently, the left member of equation (8a) can be rearranged into a single series involving $T_k(t)$. The comparison of the coefficients of $T_k(t)$

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[1]The value $Ei(a)$ may be evaluated by means of the Taylor series. In this report $Ei(a)$ is computed by first finding the Chebyshev series approximation to $[Ei(x) - \log|x| - \gamma]/x$ to get $Ei(a)$. The quantities e^a and $\log|a|$ for integral values of a may be found in existing tables.

then yields the infinite system of equations

$$\left.\begin{aligned}
&\sum_{k=0}^{\infty}{}' (-1)^k A_k^{(0)} = \phi(-1) \\
&2pA_{|k-1|}^{(1)} + 4qA_k^{(1)} + 2pA_{k+1}^{(1)} + p^2A_{|k-1|}^{(0)} + 2p(q-2)A_k^{(0)} + p^2A_{k+1}^{(0)} \\
&\qquad = \begin{cases} 4pq\ , & k = 0 \\ 2p^2\ , & k = 1 \\ 0\ , & k = 2, 3, \ldots \end{cases}
\end{aligned}\right\} \tag{10}$$

The relation (ref. 7)

$$2kA_k^{(0)} = A_{k-1}^{(1)} - A_{k+1}^{(1)} \tag{11}$$

can be used to reduce equations (10) to a system of equations involving only $A_k^{(0)}$. Thus, replacing k of equations (10) by $k+2$ and subtracting the resulting equation from equations (10), we have, by means of equation (11), the system of equations

$$\left.\begin{aligned}
&\sum_{k=0}^{\infty}{}' (-1)^k A_k = \phi(-1) \\
&2p(q-2)A_0 + (8q + p^2)A_1 + 2p(6-q)A_2 - p^2A_3 = 4pq \\
&p^2A_{k-1} + 2p(2k+q-2)A_k + 8q(k+1)A_{k+1} + 2p(2k-q+6)A_{k+2} - p^2A_{k+3} \\
&\qquad = \begin{cases} 2p^2\ , & k = 1 \\ 0\ , & k = 2, 3, \ldots \end{cases}
\end{aligned}\right\} \tag{12}$$

The superscript of $A_k^{(0)}$ is dropped for simplicity. In order to solve the infinite system (12), Clenshaw (ref. 4) essentially considered the required solution as the limiting solution of the sequence of truncated systems consisting of the first $M + 1$ equations of the same system, that is, the solution of the system

$$\sum_{k=0}^{M}{}' \; (-1)^k A_k = \phi(-1) \tag{13a}$$

$$2p(q - 2)A_0 + (8q + p^2)A_1 + 2p(q - 6)A_2 - p^2 A_3 = 4pq \tag{13b}$$

$$\left.\begin{aligned}
&p^2 A_{k-1} + 2p(2k + q - 2)A_k + 8q(k + 1)A_{k+1} + 2p(2k - q + 6)A_{k+2} - p^2 A_{k+3} \\
&\qquad = \begin{cases} 2p^2 , & k = 1 \\ 0 , & k = 2, 3, \ldots , M - 3 \end{cases} \\
&p^2 A_{M-3} + 2p(2M + q - 6)A_{M-2} + 8q(M - 1)A_{M-1} + 2p(2M + 4 - q)A_M = 0 \\
&p^2 A_{M-2} + 2p(2M + q - 4)A_{M-1} + 8qMA_M = 0
\end{aligned}\right\} \tag{13c}$$

where A_k is assumed to vanish for $K \geq M + 1$. To solve system (13), consider first the subsystem (13c) consisting of $M-2$ equations in M unknowns. Here use is made of the fact that the subsystem (13c) is satisfied by

$$A_k = c_1\alpha_k + c_2\beta_k + \gamma_k \qquad (k = 0, 1, 2, \ldots) \tag{14}$$

for arbitrary constants c_1 and c_2, where γ_k is a particular solution of (13c) and where α_k and β_k are two independent solutions of the homogeneous equations ((13c) with $2p^2$ deleted) of the same subsystem. Hence, if α_k, β_k, and γ_k are available, the solution of system (13) reduces to the determination of c_1 and c_2 from equations (13a) and (13b).

To solve equations (13), we note that

$$\gamma_0 = 2 , \qquad \gamma_k = 0 , \qquad \text{for} \quad k = 1(1)M \tag{15}$$

is obviously a particular solution of equation (13c). The two independent solutions α_k and β_k of the homogeneous equations of the same subsystem can be generated in turn by backward recurrence if we set

and
$$\left.\begin{aligned} \alpha_{M-1} &= 0 , \qquad \alpha_M = 1 \\ \beta_{M-1} &= 1 , \qquad \beta_M = 0 \end{aligned}\right\} \tag{16}$$

or choose any α_{M-1}, α_M, and β_{M-1}, β_M for which $\alpha_{M-1}\beta_M - \alpha_M\beta_{M-1} \neq 0$. The arbitrary constants c_1 and c_2 are determined, and consequently the solution of equations (13) if equation (14) is substituted into equations (13a) and (13b) and the resulting equations

$$c_1R(\alpha) + c_2R(\beta) = \phi(-1) - 1 \tag{17a}$$

$$c_1S(\alpha) + c_2S(\beta) = 8p \tag{17b}$$

are solved as two equations in two unknowns. The terms $R(\alpha)$ and $S(\alpha)$ are equal, respectively, to the left members of equations (13a) and (13b) corresponding to solution α_k. (The identical designation holds for $R(\beta)$ and $S(\beta)$.)

The quantities α_k and β_k are known as trial solutions in reference 4. Clenshaw has pointed out that if α_k and β_k are not sufficiently independent, loss of significance will occur in the formation of the linear combination (14), with a consequent loss of accuracy. Clenshaw suggested the Gauss-Seidel iteration procedure to improve the accuracy of the solution. However, this requires the application of an additional computing procedure and may prove to be extremely slow. A simpler procedure which does not alter the basic computing scheme given above is proposed here. The loss of accuracy can effectively be regained if we first generate a third trial solution δ_k $(k = 0, 1, \ldots, M)$, where δ_{M-1} and δ_M are equal to $c_1\alpha_{M-1} + c_2\beta_{M-1}$ and $c_1\alpha_M + c_2\beta_M$, respectively, and where δ_k $(k = M-2, M-3, \ldots, 0)$ is determined using backward recurrence as before by means of equations (13c). Then either α_k or β_k is replaced by δ_k and a new set of c_1 and c_2 is determined by equations (17a) and (17b). Such a procedure can be repeated until the required accuracy is reached. However, only one application of it was necessary in the computation of the coefficients of this report.

As an example, consider the case for $4 \leq x \leq 12$ with $M = 15$. The right member of equation (17a) and of equation (17b) assume, respectively, the values 0.43820800 and 64. The trial solutions α_k and β_k generated with $\alpha_{14} = 8$, $\alpha_{15} = 9$, and $\beta_{14} = 7$, $\beta_{15} = 8$ are certainly independent, since $\alpha_{14}\beta_{15} - \alpha_{15}\beta_{14} = 1 \neq 0$. A check of table 2 shows that equations (17a) and (17b) have, respectively, the residuals -0.137×10^{-4} and -0.976×10^{-3}. The same table also shows that $c_1\alpha_k$ is opposite in sign but nearly equal in magnitude to $c_2\beta_k$. Cancellations in the formation of the linear combination (14) caused a loss of significance of 2 to 6 figures in the computed A_k. In the second iteration, where a new set of β_k is generated by replacing β_{14} and β_{15}, respectively, by $c_1\alpha_{14} + c_2\beta_{14}$ and $c_1\alpha_{15} + c_2\beta_{15}$ of the first iteration, the new $c_1\alpha_k$ and $c_2\beta_k$ differed from 2 to 5 orders of magnitude. Consequently, no cancellation of significant figures in the computation of A_k occurred. Notice that equations (17) are now satisfied exactly. Further note that the new c_1 and c_2 are near zero and unity, respectively, for the reason that if equations (13) are satisfied by equation (14) exactly in the first iteration, the new c_1 and c_2 should have the precise values zero and 1, respectively. The results of the third iteration show that the A_k of the second iteration are already accurate to eight decimal places, since the A_k in the two iterations differ by less than 0.5×10^{-8}. Notice that for the third iteration, equations (17) are also satisfied exactly and that $c_1 = 1$ and $c_2 = 0$ (relative to 8 place accuracy).

TABLE 2.- COMPUTATION OF CHEBYSHEV COEFFICIENTS FOR $xe^{-x}Ei(x)$
[$4 \leq x \leq 12$ with $M = 15$; $\gamma_0 = 2$, $\gamma_k = 0$ for $k = 1(1)15$]

First iteration: $\alpha_{14} = 8$, $\alpha_{15} = 9$; $\beta_{14} = 7$, $\beta_{15} = 8$

k	$c_1\alpha_k$	$c_2\beta_k$	A_k
0	0.71690285E 03	-0.71644773E 03	0.24551200E 01
1	-0.33302683E 03	0.33286440E 03	-0.16243000E 00
2	0.13469341E 03	-0.13464845E 03	0.44960000E-01
3	-0.43211869E 02	0.43205127E 02	-0.67420000E-02
4	0.99929173E 01	-0.99942238E 01	-0.13065000E-02
5	-0.11670764E 01	0.11684574E 01	0.13810000E-02
6	-0.25552137E 00	0.25493635E 00	-0.58502000E-02
7	0.20617247E 00	-0.20599754E 00	0.17493000E-03
8	-0.75797238E-01	0.75756767E-01	-0.40471000E-04
9	0.20550680E-01	-0.20543463E-01	0.72170000E-05
10	-0.45192333E-02	0.45183721E-02	-0.86120000E-06
11	0.82656562E-03	-0.82656589E-03	-0.27000000E-09
12	-0.12333571E-03	0.12337366E-03	0.37950000E-07
13	0.13300910E-04	-0.13315328E-04	-0.14418000E-07
14	-0.29699001E-06	0.30091136E-06	0.39213500E-08
15	-0.33941716E-06	0.33852528E-06	-0.89188000E-09

$c_1 = 0.37613920E-07$

$c_2 = -0.42427144E-07$

$c_1R(\alpha) + c_2R(\beta) - 0.43820800E\ 00 = -0.13700000E-04$

$c_1S(\alpha) + c_2S(\beta) - 0.64000000E\ 02 = -0.97600000E-03$

Second iteration: $\alpha_{14} = 8$, $\alpha_{15} = 9$; $\beta_{14} = 0.39213500E-08$, $\beta_{15} = -0.89188000E-09$

k	$c_1\alpha_k$	$c_2\beta_k$	A_k
0	0.36701576E-05	0.45512986E 00	0.24551335E 01
1	-0.17051695E-05	-0.16243666E 00	-0.16243837E 00
2	0.68976566E-06	0.44956834E-01	0.44957523E-01
3	-0.22132756E-06	-0.67413538E-02	-0.67415751E-02
4	0.51197561E-07	-0.13067496E-02	-0.13066984E-02
5	-0.59856744E-08	0.13810895E-02	0.13810835E-02
6	-0.13059663E-08	-0.58502164E-03	-0.58502294E-03
7	0.10552667E-08	0.17492889E-03	0.17492994E-03
8	-0.38808033E-09	-0.40472426E-04	-0.40472814E-04
9	0.10523831E-09	0.72169965E-05	0.72171017E-05
10	-0.23146333E-10	-0.86125438E-06	-0.86127752E-06
11	0.42342615E-11	-0.25542252E-09	-0.25118825E-09
12	-0.63200810E-12	0.37946968E-07	0.37946336E-07
13	0.68210630E-13	-0.14417584E-07	-0.14417516E-07
14	-0.15414832E-14	0.39212981E-08	0.39212965E-08
15	-0.17341686E-14	-0.89186818E-09	-0.89186991E-09

$c_1 = -0.19268540E-15$

$c_2 = 0.99998675E\ 00$

$c_1R(\alpha) + c_2R(\beta) - 0.43820800E\ 00 = 0.0$

$c_1S(\alpha) + c_2S(\beta) - 0.64000000E\ 02 = 0.0$

TABLE 2.- COMPUTATION OF CHEBYSHEV COEFFICIENTS FOR $xe^{-x}Ei(x)$ - Concluded
[$4 \le x \le 12$ with $M = 15$; $\gamma_0 = 2$, $\gamma_k = 0$ for $k = 1(1)15$]

Third iteration: $\alpha_{14} = 8$, $\alpha_{15} = 9$; $\beta_{14} = 0.39212965E-08$, $\beta_{15} = -0.89186991E-09$

k	$c_1\alpha_k$	$c_2\beta_k$	A_k
0	-0.23083059E-07	0.45513355E 00	0.24551335E 01
1	0.10724479E-07	-0.16243838E 00	-0.16243837E 00
2	-0.43382065E-08	0.44957526E-01	0.44957522E-01
3	0.13920157E-08	-0.67415759E-02	-0.67415745E-02
4	-0.32200152E-09	-0.13066983E-02	-0.13066986E-02
5	0.37646251E-10	0.13810835E-02	0.13810836E-02
6	0.82137336E-11	-0.58502297E-03	-0.58502296E-03
7	-0.66369857E-11	0.17492995E-03	0.17492994E-03
8	0.24407892E-11	-0.40472817E-04	-0.40472814E-04
9	-0.66188494E-12	0.72171023E-05	0.72171017E-05
10	0.14557636E-12	-0.86127766E-06	-0.86127751E-06
11	-0.26630930E-13	-0.25116620E-09	-0.25119283E-09
12	0.39749465E-14	0.37946334E-07	0.37946337E-07
13	-0.42900337E-15	-0.14417516E-07	-0.14417516E-07
14	0.96949915E-17	0.39212966E-08	0.39212966E-08
15	0.10906865E-16	-0.89186992E-09	-0.89186990E-09

$$c_1 = 0.12118739E-17$$

$$c_2 = 0.10000000E\ 01$$

$$c_1R(\alpha) + c_2R(\beta) - 0.43820800E\ 00 = 0.0$$

$$c_1S(\alpha) + c_2S(\beta) - 0.64000000E\ 02 = 0.0$$

It is worth noting that the coefficient matrix of system (13) yields an upper triangular matrix of order $M - 1$ after the deletion of the first two rows and the last two columns. Consequently, the procedure of this section is applicable to any linear system having this property. As a matter of fact, the same procedure can be generalized to solve linear systems having coefficient matrices of order N, the deletion of whose first r $(r < N)$ rows and last r columns yields upper triangular matrices of order $N - r$.

The Function $(1/x)[Ei(x) - \log|x| - \gamma]$

Let

$$f(x) = (1/x)[Ei(x) - \log|x| - \gamma], \qquad g(x) = e^x, \qquad |x| \le b \tag{18}$$

These functions, with the change of variable $x = bt$, simultaneously satisfy the differential equations

$$bt^2\phi'(t) + bt\phi(t) - \psi(t) = -1 \tag{19a}$$

$$\psi'(t) - b\psi(t) = 0, \qquad -1 \le t \le 1 \tag{19b}$$

Conversely,[2] any solution of equations (19) is equal to the functions given by equations (18) for the change of variable $x = bt$. Therefore, boundary conditions need not be imposed for the solution of the differential equations.

A procedure similar to that of the previous section gives the coupled infinite recurrence relations

$$bA_1 + bA_3 - B_0 + B_2 = -2 \tag{20a}$$

$$\left.\begin{aligned} kbA_{k-1} + 2(k + 1)bA_{k+1} + (k + 2)bA_{k+3} - 2B_k + 2B_{k+2} &= 0 \\ bB_{k-1} - 2kB_k - bB_{k+1} = 0 , \qquad k = 1, 2, \ldots & \end{aligned}\right\} \tag{20b}$$

where A_k and B_k are the Chebyshev coefficients of $\phi(t)$ and $\psi(t)$, respectively.

Consider first the subsystem (20b). If $A_k = \alpha_k$ and $B_k = \beta_k$ are a simultaneous solution of the subsystem, which is homogeneous, then

$$\left.\begin{aligned} A_k &= c\alpha_k \\ \text{and} \qquad B_k &= c\beta_k \end{aligned}\right\} \tag{21}$$

are also a solution for an arbitrary constant c. Thus based on considerations analogous to the solution of equations (13), one can initiate an approximate solution of equations (20) by setting

$$\left.\begin{aligned} \alpha_M &= 0 , \qquad \alpha_k = 0 \qquad \text{for} \quad k \geq M + 1 \\ \beta_M &= 1 , \qquad \beta_k = 0 \qquad \text{for} \quad k \geq M + 1 \end{aligned}\right\} \tag{22}$$

and then determining α_k and β_k $(k = M - 1, M - 2, \ldots, 0)$ by backward recurrence by means of equation (20b). The arbitrary constant c is determined by substituting equations (21) in equation (20a).

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[2]The general solution of the differential equations has the form

$$\phi(t) = (c_1/t) + [\mathrm{Ei}(bt) - \log |bt| - \gamma]/bt$$

$$\psi(t) = c_2 e^{bt}$$

where the first and second terms of $\phi(t)$ are, respectively, the complementary solution and a particular integral of equation (19a). The requirement that $\phi(t)$ is bounded makes the constant $c_1 = 0$. The fact that $\psi(0) = 1$ is implicit in equation (19a).

The Function $xe^{-x}Ei(x)$ on the Infinite Interval

Let

$$f(x) = xe^{-x}Ei(x) , \quad -\infty < x \leq b < 0 , \quad \text{or} \quad 0 < b \leq x < \infty \tag{23}$$

By making the change of variables,

$$x = 2b/(t + 1) \tag{24}$$

we can easily demonstrate that

$$f(x) = f[2b/(t + 1)] = \phi(t) \tag{25}$$

satisfies the differential equation

$$(t + 1)^2\phi'(t) + (t + 1 - 2b)\phi(t) = -2b \tag{26a}$$

with

$$\phi(1) = be^{-b}Ei(b) \tag{26b}$$

An infinite system of equations involving the Chebyshev coefficients A_k of $\phi(t)$ is deducible from equations (26) by the same procedure as applied to equations (7) to obtain the infinite system (12); it is given as follows.

$$\sum_{k=0}^{\infty}{}' A_k = \phi(1) = be^{-b}Ei(b) \tag{27a}$$

$$(1 - 2b)A_0 + 3A_1 + (3 + 2b)A_2 + A_3 = -4b \tag{27b}$$

$$kA_{k-1} + 2[(2k + 1) - 2b]A_k + 6(k + 1)A_{k+1} + 2(2k + 3 + 2b)A_{k+2}$$
$$+ (k + 2)A_{k+3} = 0 , \quad k = 1, 2, . . . \tag{27c}$$

As in the case of equations (13), the solution of equations (27) can be assumed to be

$$A_k = c_1\alpha_k + c_2\beta_k \tag{28}$$

with A_k vanishing for a $k \geq M$. Thus, we can set, say

$$\left.\begin{aligned} \alpha_{M-1} &= 0 , & \alpha_M &= 1 \\ \beta_{M-1} &= 1 , & \beta_M &= 0 \end{aligned}\right\} \tag{29}$$

and determine the trial solutions α_k and β_k $(k = M - 1, M - 2, \ldots, 0)$ by means of equation (27c) by backward recurrence. The required solution of equations (27) is then determined by substituting equation (28) in equations (27a) and (27b) and solving the resulting equations for c_1 and c_2.

Loss of accuracy in the computation of A_k can also occur here, as in the solution of the system (13), if the trial solutions are not sufficiently independent. The process used to improve the accuracy of A_k of the system (13) can also be applied here.

For efficiency in computation, it is worth noting that for $b < 0$ $(-\infty < x \leq b < 0)$ the boundary condition (26b) is not required for the solution of equations (26). This follows from the fact that any solution[3] of the differential equation (26a) is equal to $xe^{-x}Ei(x)$ $(x = 2b/(t + 1))$. Hence the A_k of $xe^{-x}Ei(x)$ for $-\infty < x \leq b < 0$ can be obtained without the use of equation (27a) and can be assumed to have the form

$$A_k = c\alpha_k , \qquad (k = 0, 1, \ldots, M) \tag{30}$$

The $M + 1$ values of α_k can be generated by setting $\alpha_M = 1$ and computing α_k $(k = 0, 1, \ldots, M-1)$ by means of equation (27c) by backward recurrence. The substitution of equation (30) in equation (27b) then enables one to determine c from the resulting equation.

REMARKS ON CONVERGENCE AND ACCURACY

The Chebyshev coefficients of table 3 were computed on the IBM 7094 with 50-digit normalized floating-point arithmetic. In order to assure that the sequence of approximate solutions (see Discussion) converged to the limiting solution of the differential equation in question, a trial M was incremented by 4 until the approximate Chebyshev coefficients showed no change greater than or equal to 0.5×10^{-35}. Hence the maximum error is bounded by

$$0.5(M + 1) \times 10^{-35} + \sum_{M+1}^{\infty} |A_k|$$

---

[3]The general solution of the differential equation (26a) is of the form

$$\phi(t) = cxe^{-x} + xe^{-x}Ei(x) , \qquad x = 2b/(t + 1)$$

where the first and second terms are equal, respectively, to the complementary solution and a particular integral of equation (26a). Since equation (26a) has no bounded complementary solution for $-\infty < x \leq b < 0$, every solution of it is equal to the particular integral $xe^{-x}Ei(x)$. On the other hand, a solution of equation (26a) for $0 < x \leq b < \infty$ would, in general, involve the complementary function. Hence, boundary condition (26b) is required to guarantee that the solution of equation (26a) is equal to $xe^{-x}Ei(x)$.

where the first term is the maximum error of the $M + 1$ approximate Chebyshev coefficients, and the sum is the maximum error of the truncated Chebyshev series of $M + 1$ terms. If the Chebyshev series is rapidly convergent, the maximum error of the approximate Chebyshev series should be of the order of 10^{-30}. The coefficients of table 3 have been rounded to 30 digits, and higher terms for $k > N$ giving the maximum residual

$$\sum_{k=N+1}^{M} |A_k| < 0.5\times10^{-30}$$

have been dropped. This should allow for evaluation of the relevant function that is accurate to 30 decimal places. Since the range of values of each function is bounded between 2/5 and 5, the evaluated function should be good to 30 significant digits. Taylor series evaluation also checks with that of the function values of table 4 (computed with 30-digit floating-point arithmetic using the coefficients of table 3) for at least 28-1/2 significant digits. Evaluation of $Ei(x)$ using the coefficients of table 3 also checked with Murnaghan and Wrench (ref. 6) for 28-1/2 significant figures.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., 94035, April 6, 1970

TABLE 3.- CHEBYSHEV COEFFICIENTS

(a) $xe^{-x}Ei(x) = \sum_{k=0}^{40}{}' A_k T_k(t), \; t = (-20/x) - 1, (-\infty < x \leq -10)$

k	A_k
0	0.1912173225 8605534539 1519326510 E 01
1	-0.4208355052 8684843755 0974986680 E-01
2	0.1722819627 2843267833 7118157835 E-02
3	-0.9915782173 4445636455 9842322973 E-04
4	0.7176093168 0227750526 5590665592 E-05
5	-0.6152733145 0951269682 7956791331 E-06
6	0.6024857106 5627583129 3999701610 E-07
7	-0.6573848845 2883048229 5894189637 E-08
8	0.7853167541 8323998199 4810079871 E-09
9	-0.1013730288 0038789855 4202774257 E-09
10	0.1399770413 2267686027 7823488623 E-10
11	-0.2051008376 7838189961 8962318711 E-11
12	0.3168388726 0024778181 4907985818 E-12
13	-0.5132760082 8391806541 5984751899 E-13
14	0.8680933040 7665493418 7433687383 E-14
15	-0.1527015040 9030849719 8572355351 E-14
16	0.2784686251 6493573965 0105251453 E-15
17	-0.5249890437 4217669680 8472933696 E-16
18	0.1020717991 2485612924 7455787226 E-16
19	-0.2042264679 8997184130 8462421876 E-17
20	0.4197064172 7264847440 8827228562 E-18
21	-0.8844508176 1728105081 6483737536 E-19
22	0.1908272629 5947174199 5060168262 E-19
23	-0.4209746222 9351995033 6450865676 E-20
24	0.9483904058 1983732764 1500214512 E-21
25	-0.2179467860 1366743199 4032574014 E-21
26	0.5103936869 0714509499 3452562741 E-22
27	-0.1216883113 3344150908 9746779693 E-22
28	0.2951289166 4478751929 4773757144 E-23
29	-0.7275353763 7728468971 4438950920 E-24
30	0.1821639048 6230739612 1667115976 E-24
31	-0.4629629963 1633171661 2753482064 E-25
32	0.1193539790 9715779152 3052371292 E-25
33	-0.3119493285 2201424493 1062147473 E-26
34	0.8261419734 5334664228 4170028518 E-27
35	-0.2215803373 6609829830 2591177697 E-27
36	0.6016031671 6542638904 5303124429 E-28
37	-0.1652725098 3821265964 9744302314 E-28
38	0.4592230358 7730270279 5636377166 E-29
39	-0.1290062767 2132638473 7453212670 E-29
40	0.3662718481 0320025908 1177078922 E-30

TABLE 3.- CHEBYSHEV COEFFICIENTS - Continued

$$\text{(b)}\quad xe^{-x}\mathrm{Ei}(x) = \sum_{k=0}^{43}{}' A_k T_k(t),\ t = (x + 7)/3,\ (-10 \le x \le -4)$$

k	A_k			
0	0.1757556496	0612937384	8762834691	E 01
1	-0.4358541517	7361661170	5001867964	E-01
2	-0.7979507139	5584254013	3217027492	E-02
3	-0.1484372327	3037121385	0970210001	E-02
4	-0.2800301984	3775145748	6203954948	E-03
5	-0.5348648512	8657932303	9177361553	E-04
6	-0.1032867243	5735548661	0233266460	E-04
7	-0.2014083313	0055368773	2226198639	E-05
8	-0.3961758434	2738664582	2338443500	E-06
9	-0.7853872767	0966316306	7607656069	E-07
10	-0.1567925981	0074698262	4616270279	E-07
11	-0.3150055939	3763998825	0007372851	E-08
12	-0.6365096822	5242037304	0380263972	E-09
13	-0.1292888113	2805631835	6593121259	E-09
14	-0.2638690999	6592557613	2149942808	E-10
15	-0.5408958287	0450687349	1922207896	E-11
16	-0.1113222784	6010898999	7676692708	E-11
17	-0.2299624726	0744624618	4338864145	E-12
18	-0.4766682389	4951902622	3913482091	E-13
19	-0.9911756747	3352709450	6246643371	E-14
20	-0.2067103580	4957072400	0900805021	E-14
21	-0.4322776783	3833850564	5764394579	E-15
22	-0.9063014799	6650172551	4905603356	E-16
23	-0.1904669979	5816613974	4015963342	E-16
24	-0.4011792326	3502786634	6744227520	E-17
25	-0.8467772130	0168322313	4166334685	E-18
26	-0.1790842733	6586966555	5826492204	E-18
27	-0.3794490638	1714782440	1106175166	E-19
28	-0.8053999236	7982798526	0999654058	E-20
29	-0.1712339011	2362012974	3228671244	E-20
30	-0.3646274058	7749686208	6576562816	E-21
31	-0.7775969638	8939479435	3098157647	E-22
32	-0.1660628498	4484020566	2531950966	E-22
33	-0.3551178625	7882509300	5927145352	E-23
34	-0.7603722685	9413580929	5734653294	E-24
35	-0.1630074137	2584900288	9638374755	E-24
36	-0.3498575202	7286322350	7538497255	E-25
37	-0.7517179627	8900988246	0645145143	E-26
38	-0.1616877440	0527227629	8777317918	E-26
39	-0.3481270085	7247569174	8202271565	E-27
40	-0.7502707775	5024654701	0642233720	E-28
41	-0.1618454364	4959102680	7612330206	E-28
42	-0.3494366771	7051616674	9482836452	E-29
43	-0.7551036906	1261678585	6037026797	E-30

TABLE 3.- CHEBYSHEV COEFFICIENTS - Continued

(c) $[Ei(x) - \log|x| - \gamma]/x = \sum_{k=0}^{33}{}' A_k T_k(t)$, $t = x/4$, $(-4 \le x \le 4)$

k	A_k
0	0.3293700103 7673912939 3905231421 E 01
1	0.1679835052 3713029156 5505796064 E 01
2	0.7220436105 6787543524 0299679644 E 00
3	0.2600312360 5480956171 3740181192 E 00
4	0.8010494308 1737502239 4742889237 E-01
5	0.2151403663 9763337548 0552483005 E-01
6	0.5116207789 9303312062 1968910894 E-02
7	0.1090932861 0073913560 5066199014 E-02
8	0.2107415320 2393891631 8348675226 E-03
9	0.3719904516 6518885709 5940815956 E-04
10	0.6043491637 1238787570 4767032866 E-05
11	0.9092954273 9626095264 9596541772 E-06
12	0.1273805160 6592647886 5567184969 E-06
13	0.1669185748 4109890739 0896143814 E-07
14	0.2054417026 4010479254 7612484551 E-08
15	0.2383584444 4668176591 4052321417 E-09
16	0.2615386378 8854429666 9068664148 E-10
17	0.2721858622 8541670644 6550268995 E-11
18	0.2693750031 9835792992 5326427442 E-12
19	0.2541220946 7072635546 7884089307 E-13
20	0.2290130406 8650370941 8510620516 E-14
21	0.1975465739 0746229940 1057650412 E-15
22	0.1634024551 9289317406 8635419984 E-16
23	0.1298235437 0796376099 1961293204 E-17
24	0.9922587925 0737105964 4632581302 E-19
25	0.7306252806 7221032944 7230880087 E-20
26	0.5189676834 6043451272 0780080019 E-21
27	0.3560409454 0997068112 8043162227 E-22
28	0.2361979432 5793864237 0187203948 E-23
29	0.1516837767 7214529754 9624516819 E-24
30	0.9439089722 2448744292 5310405245 E-26
31	0.5697227559 5036921198 9581737831 E-27
32	0.3338333627 7954330315 6597939562 E-28
33	0.1900626012 8161914852 6680482237 E-29

(γ = 0.5772156649 0153286060 6512090082 E 00)

TABLE 3.- CHEBYSHEV COEFFICIENTS - Continued

(d) $xe^{-x}Ei(x) = \sum_{k=0}^{49}{}' A_k T_k(t), \; t = (x - 8)/4, \; (4 \le x \le 12)$

k	A_k
0	0.2455133538 7812952867 3420457043 E 01
1	-0.1624383791 3037652439 6002276856 E 00
2	0.4495753080 9357264148 0785417193 E-01
3	-0.6741578679 9892299884 8718835050 E-02
4	-0.1306697142 8032942805 1599341387 E-02
5	0.1381083146 0007257602 0202089820 E-02
6	-0.5850228790 1596579868 7368242394 E-03
7	0.1749299341 0789197003 8740976432 E-03
8	-0.4047281499 0529303552 2869333800 E-04
9	0.7217102412 1709975003 5752600049 E-05
10	-0.8612776970 1986775241 4815450193 E-06
11	-0.2514475296 5322559777 9084739054 E-09
12	0.3794747138 2014951081 4074505574 E-07
13	-0.1442117969 5211980616 0265640172 E-07
14	0.3935049295 9761013108 7190848042 E-08
15	-0.9284689401 0633175304 7289210353 E-09
16	0.2031789568 0065461336 6090995698 E-09
17	-0.4292498504 9923683142 7918026902 E-10
18	0.8992647177 7812393526 8001544182 E-11
19	-0.1900869118 4121097524 2396635722 E-11
20	0.4092198912 2237383452 6121178338 E-12
21	-0.8999253437 2931901982 5435824585 E-13
22	0.2019654670 8242638335 4948543451 E-13
23	-0.4612930261 3830820719 4950531726 E-14
24	0.1069023072 9386369566 8857256409 E-14
25	-0.2507030070 5700729569 2572254042 E-15
26	0.5937322503 7915516070 6073763509 E-16
27	-0.1417734582 4376625234 4732005648 E-16
28	0.3409203754 3608089342 6806402093 E-17
29	-0.8248290269 5054937928 8702529656 E-18
30	0.2006369712 6214423139 8824095937 E-18
31	-0.4903851667 9674222440 3498152027 E-19
32	0.1203734482 3483321716 6664609324 E-19
33	-0.2966282447 1413682538 1453572575 E-20
34	0.7335512384 2880759924 2142328436 E-21
35	-0.1819924142 9085112734 4263485604 E-21
36	0.4528629374 2957606021 7359526404 E-22
37	-0.1129980043 7506096133 8906717853 E-22
38	0.2826681251 2901165692 3764408445 E-23
39	-0.7087717977 1690496166 6732640699 E-24
40	0.1781104524 0187095153 4401530034 E-24
41	-0.4485004076 6189635731 2006142358 E-25
42	0.1131540292 5754766224 5053090840 E-25
43	-0.2859957899 7793216379 0414326136 E-26
44	0.7240775806 9226736175 8172726753 E-27
45	-0.1836132234 1257789805 0666710105 E-27
46	0.4663128735 2273048658 2600122073 E-28
47	-0.1185959588 9190288794 6724005478 E-28
48	0.3020290590 5567131073 1137614875 E-29
49	-0.7701650548 1663660609 8827057102 E-30

TABLE 3.- CHEBYSHEV COEFFICIENTS - Continued

(e) $$xe^{-x}Ei(x) = \sum_{k=0}^{47}{}' A_k T_k(t), \quad t = (x - 22)/10, \quad (12 \le x \le 32)$$

k	A_k
0	0.2117028640 4369866832 9789991614 E 01
1	-0.3204237273 7548579499 0618303177 E-01
2	0.8891732077 3531683589 0182400335 E-02
3	-0.2507952805 1892993708 8352442063 E-02
4	0.7202789465 9598754887 5760902487 E-03
5	-0.2103490058 5011305342 3531441256 E-03
6	0.6205732318 2769321658 8857730842 E-04
7	-0.1826566749 8167026544 9155689733 E-04
8	0.5270651575 2893637500 7788296811 E-05
9	-0.1459666547 6199457532 3066719367 E-05
10	0.3781719973 5896367198 0484193981 E-06
11	-0.8842581282 8407192007 7971589012 E-07
12	0.1741749198 5383936137 7350309156 E-07
13	-0.2313517747 0436906350 6474480152 E-08
14	-0.1228609819 1808623883 2104835230 E-09
15	0.2349966236 3228637047 8311381926 E-09
16	-0.1100719401 0272628769 0738963049 E-09
17	0.3848275157 8612071114 9705563369 E-10
18	-0.1148440967 4900158965 8439301603 E-10
19	0.3056876293 0885208263 0893626200 E-11
20	-0.7388278729 2847356645 4163131431 E-12
21	0.1630933094 1659411056 4148013749 E-12
22	-0.3276989373 3127124965 7111774748 E-13
23	0.5898114347 0713196171 1164283918 E-14
24	-0.9099707635 9564920464 3554720718 E-15
25	0.1040752382 6695538658 5405697541 E-15
26	-0.1809815426 0592279322 7163355935 E-17
27	-0.3777098842 5639477336 9593494417 E-17
28	0.1580332901 0284795713 6759888420 E-17
29	-0.4684291758 8088273064 8433752957 E-18
30	0.1199516852 5919809370 7533478542 E-18
31	-0.2823594749 8418651767 9349931117 E-19
32	0.6293738065 6446352262 7520190349 E-20
33	-0.1352410249 5047975630 5343973177 E-20
34	0.2837106053 8552914159 0980426210 E-21
35	-0.5867007420 2463832353 1936371015 E-22
36	0.1205247636 0954731111 2449686917 E-22
37	-0.2474446616 9988486972 8416011246 E-23
38	0.5099962585 8378500814 2986465688 E-24
39	-0.1058382578 7754224088 7093294733 E-24
40	0.2215276245 0704827856 6429387155 E-25
41	-0.4679278754 7569625867 1852546231 E-26
42	0.9972872990 6020770482 4269828079 E-27
43	-0.2143267945 2167880459 1907805844 E-27
44	0.4640656908 8381811433 8414829515 E-28
45	-0.1011447349 2115139094 8461800780 E-28
46	0.2217211522 7100771109 3046878345 E-29
47	-0.4884890469 2437855322 4914645512 E-30

TABLE 3.- CHEBYSHEV COEFFICIENTS - Concluded

(f) $xe^{-x}Ei(x) = \sum_{k=0}^{46}{}' A_k T_k(t)$, $t = (64/x) - 1$, $(32 \le x < \infty)$

k	A_k
0	0.2032843945 7961669908 7873844202 E 01
1	0.1669920452 0313628514 7618434339 E-01
2	0.2845284724 3613468074 2489985325 E-03
3	0.7563944358 5162064894 8786693854 E-05
4	0.2798971289 4508591575 0484318090 E-06
5	0.1357901828 5345310695 2556392593 E-07
6	0.8343596202 0404692558 5610289412 E-09
7	0.6370971727 6402484382 7524337306 E-10
8	0.6007247608 8118612357 6083084850 E-11
9	0.7022876174 6797735907 5059216588 E-12
10	0.1018302673 7036876930 9667322152 E-12
11	0.1761812903 4308800404 0656741554 E-13
12	0.3250828614 2353606942 4072007647 E-14
13	0.5071770025 5058186788 1479300685 E-15
14	0.1665177387 0432942985 3520036957 E-16
15	-0.3166753890 7975144007 2410018963 E-16
16	-0.1588403763 6641415154 8423134074 E-16
17	-0.4175513256 1380188308 9626455063 E-17
18	-0.2892347749 7071418820 2868862358 E-18
19	0.2800625903 3966080728 9978777339 E-18
20	0.1322938639 5392708914 0532005364 E-18
21	0.1804447444 1773019958 5334811191 E-19
22	-0.7905384086 5226165620 2021080364 E-20
23	-0.4435711366 3695734471 8167314045 E-20
24	-0.4264103994 9781026176 0579779746 E-21
25	0.3920101766 9371439072 5625388636 E-21
26	0.1527378051 3439636447 2804486402 E-21
27	-0.1024849527 0494906078 6953149788 E-22
28	-0.2134907874 7710893794 8904287231 E-22
29	-0.3239139475 1602368761 4279789345 E-23
30	0.2142183762 2964597029 6249355934 E-23
31	0.8234609419 6189955316 9207838151 E-24
32	-0.1524652829 6206721081 1495038147 E-24
33	-0.1378208282 4882440129 0438126477 E-24
34	0.2131311201 4287370679 1513005998 E-26
35	0.2012649651 8713266585 9213006507 E-25
36	0.1995535662 0563740232 0607178286 E-26
37	-0.2798995812 2017971142 6020884464 E-26
38	-0.5534511830 5070025094 9784942560 E-27
39	0.3884995422 6845525312 9749000696 E-27
40	0.1121304407 2330701254 0043264712 E-27
41	-0.5566568286 7445948805 7823816866 E-28
42	-0.2045482612 4651357628 8865878722 E-28
43	0.8453814064 4893808943 7361193598 E-29
44	0.3565755151 2015152659 0791715785 E-29
45	-0.1383652423 4779775181 0195772006 E-29
46	-0.6062142653 2093450576 7865286306 E-30

TABLE 4.- FUNCTION VALUES OF THE ASSOCIATED FUNCTIONS

x	t = -(20/x) - 1	$xe^{-x}Ei(x)$
-INF	-1.000	0.1000000000 0000000000 0000000000 E 01
-160	-0.875	0.9938266956 7406127387 8797850088 E 00
-80	-0.750	0.9878013330 9428877356 4522608410 E 00
-53 1/3	-0.625	0.9819162901 4319443961 7735426105 E 00
-40	-0.500	0.9761646031 8514305080 8000604060 E 00
-32	-0.375	0.9705398840 7466392046 2584664361 E 00
-26 2/3	-0.250	0.9650362511 2337703576 3536593528 E 00
-22 6/7	-0.125	0.9596482710 7936727616 5478970820 E 00
-20	-0.000	0.9543709099 1921683397 5195829433 E 00
-17 7/9	0.125	0.9491994907 7974574460 6445346803 E 00
-16	0.250	0.9441296577 3690297898 4149471583 E 00
-14 6/11	0.375	0.9391573444 1928424124 0422409988 E 00
-13 1/3	0.500	0.9342787466 5341046480 9375801650 E 00
-12 4/13	0.625	0.9294902984 9721403772 5319679042 E 00
-11 3/7	0.750	0.9247886511 4084169605 5993585492 E 00
-10 2/3	0.875	0.9201706542 4944567620 2148012149 E 00
-10	1.000	0.9156333393 9788081876 0698157666 E 00

x	t = (x + 7)/3	$xe^{-x}Ei(x)$
-10.000	-1.000	0.9156333393 9788081876 0698157661 E 00
-9.625	-0.875	0.9128444614 6799341885 6575662217 E 00
-9.250	-0.750	0.9098627515 2542413937 8954274597 E 00
-8.875	-0.625	0.9066672706 5475388033 4995756418 E 00
-8.500	-0.500	0.9032339019 7320784414 4682926135 E 00
-8.125	-0.375	0.8995347176 8847383630 1415777697 E 00
-7.750	-0.250	0.8955371870 8753915717 9475513219 E 00
-7.375	-0.125	0.8912031763 2125431626 7087476258 E 00
-7.000	-0.000	0.8864876725 3642935289 3993846569 E 00
-6.625	0.125	0.8813371384 6821020039 4305706270 E 00
-6.250	0.250	0.8756873647 8846593227 6462155532 E 00
-5.875	0.375	0.8694606294 5411341030 2047153364 E 00
-5.500	0.500	0.8625618846 9070142209 0918986586 E 00
-5.125	0.625	0.8548735538 9019954239 2425567234 E 00
-4.750	0.750	0.8462482991 0358736117 1665798810 E 00
-4.375	0.875	0.8364987545 5629874174 2152267582 E 00
-4.000	1.000	0.8253825996 0422333240 8183035504 E 00

TABLE 4.- FUNCTION VALUES OF THE ASSOCIATED FUNCTIONS - Continued

x	t = x/4	[Ei(x) - log |x| - γ]/x
-4.0	-1.000	0.4918223446 0781809647 9962798267 E 00
-3.5	-0.875	0.5248425066 4412835691 8258753311 E 00
-3.0	-0.750	0.5629587782 2127986313 8086024270 E 00
-2.5	-0.625	0.6073685258 5838306451 4266925640 E 00
-2.0	-0.500	0.6596316780 8476964479 5492023380 E 00
-1.5	-0.375	0.7218002369 4421992965 7623030310 E 00
-1.0	-0.250	0.7965995992 9705313428 3675865540 E 00
-0.5	-0.125	0.8876841582 3549672587 2151815870 E 00
0.0	-0.000	0.1000000000 0000000000 0000000000 E 01
0.5	0.125	0.1140302841 0431720574 6248768807 E 01
1.0	0.250	0.1317902151 4544038948 6000884424 E 01
1.5	0.375	0.1545736450 7467337302 4859074039 E 01
2.0	0.500	0.1841935755 2702059966 7788045934 E 01
2.5	0.625	0.2232103799 1211651144 5340506423 E 01
3.0	0.750	0.2752668205 6852580020 0219289740 E 01
3.5	0.875	0.3455821531 9301241243 7300898811 E 01
4.0	1.000	0.4416841111 0086991358 0118598668 E 01

x	t = (x - 8)/4	$xe^{-x}Ei(x)$
4.0	-1.000	0.1438208031 4544827847 0968670330 E 01
4.5	-0.875	0.1396419029 6297460710 0674523183 E 01
5.0	-0.750	0.1353831277 4552859779 0189174047 E 01
5.5	-0.625	0.1314143565 7421192454 1219816991 E 01
6.0	-0.500	0.1278883860 4895616189 2314099578 E 01
6.5	-0.375	0.1248391155 0017014864 0741941387 E 01
7.0	-0.250	0.1222408052 3605310590 3656846622 E 01
7.5	-0.125	0.1200421499 5996307864 3879158950 E 01
8.0	-0.000	0.1181847986 9872079731 7739362644 E 01
8.5	0.125	0.1166126525 8117484943 9918142965 E 01
9.0	0.250	0.1152759208 7089248132 2396814952 E 01
9.5	0.375	0.1141323475 9526242015 5338560641 E 01
10.0	0.500	0.1131470204 7341077803 4051681355 E 01
10.5	0.625	0.1122915570 0177606064 2888630755 E 01
11.5	0.750	0.1115430938 9980384416 4779434229 E 01
11.5	0.875	0.1108832926 3050773058 6855234934 E 01
12.0	1.000	0.1102974544 9067590726 7241234953 E 01

TABLE 4.- FUNCTION VALUES OF THE ASSOCIATED FUNCTIONS - Concluded

x	t = (x - 22)/10	$xe^{-x}Ei(x)$
12.00	-1.000	0.1102974544 9067590726 7241234952 E 01
13.25	-0.875	0.1090844898 2154756926 6468614954 E 01
14.50	-0.750	0.1081351395 7351912850 6346643795 E 01
15.75	-0.625	0.1073701384 1997572371 2157900374 E 01
17.00	-0.500	0.1067393691 9585378312 9572196197 E 01
18.25	-0.375	0.1062096608 6221502426 8372647556 E 01
19.50	-0.250	0.1057581342 1587250319 5393949410 E 01
20.75	-0.125	0.1053684451 2894094408 2102194964 E 01
22.00	-0.000	0.1050285719 6851897941 1780664532 E 01
23.25	0.125	0.1047294551 7053248581 1492365591 E 01
24.50	0.250	0.1044641267 9046436368 9761075289 E 01
25.75	0.375	0.1042271337 2023202388 5710928048 E 01
27.00	0.500	0.1040141438 3230104381 3713899754 E 01
28.25	0.625	0.1038216700 3601458768 0056548394 E 01
29.50	0.750	0.1036468726 2924118457 5154685419 E 01
30.75	0.875	0.1034874149 8964796947 2990938990 E 01
32.00	1.000	0.1033413564 2162410494 3493552567 E 01

x	t = (64/x) - 1	$xe^{-x}Ei(x)$
INF	-1.000	0.1000000000 0000000000 0000000001 E 01
512	-0.875	0.1001960799 4507119253 1337468473 E 01
256	-0.750	0.1003937130 9056986278 8009078297 E 01
170 2/3	-0.625	0.1005929275 6929291129 4663030932 E 01
128	-0.500	0.1007937524 4081401828 1776821694 E 01
102 2/5	-0.375	0.1009962177 4064497557 4367545570 E 01
85 1/3	-0.250	0.1012003545 3329884820 1864466702 E 01
73 1/7	-0.125	0.1014061949 6969713314 5942329335 E 01
64	-0.000	0.1016137723 4943253217 0357100831 E 01
56 8/9	0.125	0.1018231211 8848326968 2337017143 E 01
51 1/5	0.250	0.1020342772 9307837748 7217829808 E 01
46 6/11	0.375	0.1022472778 4054205959 1275364791 E 01
42 2/3	0.500	0.1024621614 6810783910 1187804247 E 01
39 5/13	0.625	0.1026789683 7090285245 0984510823 E 01
36 4/7	0.750	0.1028977404 1058080086 3378435059 E 01
34 2/15	0.875	0.1031185212 3646592635 5875784663 E 01
32	1.000	0.1033413564 2162410494 3493552567 E 01

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